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Direct Factors of Sylow 2-Subgroups. II

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1. INTRODUCTION

Suppose G is a finite group and G has a Sylow 2-subgroup of the form $Q \times R$. In a previous paper [4], we showed that, under certain conditions on Q (in particular, when Q is a generalized quaternion group), the group G cannot be simple. In this paper, we use some recent results of D. Goldschmidt to determine some weaker conditions on Q which guarantee that G is not simple.

Our main results are the following.

THEOREM A. *Suppose S is a Sylow 2-subgroup of a finite group G, Q and R are subgroups of S, and $S = Q \times R$. Assume that Q is indecomposable and that $\Omega_1(Q) \subseteq Z(Q)$. Assume also that Q is neither an Abelian group nor a generalized quaternion group. Then there exist subgroups Q^* and R^* of S such that*

$$S = Q^* \times R = Q \times R^* = Q^* \times R^* \tag{A1}$$

and

$$\begin{aligned}\Omega_1(Q^*) \cap O_{2',2}(G) &= \Omega_1(Q^*) \cap O_{2',2}(N_G(R^*)) \\ &= \Omega_1(Q^*) \cap O_{2',2}(C_G(R^*)).\end{aligned} \tag{A2}$$

THEOREM B. *Let Q be an indecomposable, non-Abelian finite 2-group such that $\Omega_1(Q) \subseteq Z(Q)$. Then the following conditions on Q are equivalent.*

(B1) *Whenever Q is a Sylow 2-subgroup of a finite group G, then $Q \cap O_{2',2}(G) \neq 1$.*

(B2) *Whenever R is a 2-group and $Q \times R$ is a Sylow 2-subgroup of a finite group G, then $(Q \times R) \cap O_{2',2}(G) \neq 1$.*

Here, $O_2(G)$ is defined to be the largest normal 2-subgroup of a group G, and $O_{2',2}(G)$ is defined to be the normal subgroup H of G for which H contains $O_{2'}(G)$ and $H/O_{2'}(G) = O_2(G/O_{2'}(G))$. The rest of our notation is taken from [4], to which we will refer as I.

Theorem B was conjectured to the author by D. Goldschmidt. It is an analogue of Theorem D of I. The following is an example of the difference between the two results. Take any odd integer n greater than 3 and let $q = 2^n$. Let θ be a nonidentity automorphism of the Galois field of order q such that $x\theta^2 \neq x$ for some element x of the field. Define Q to be the "Suzuki 2-group" $A(n, \theta)$, as defined in [8]. Then Q is indecomposable. Moreover, Q is not a Sylow 2-subgroup of a simple group in Suzuki's family (defined in [10]), by [8, Theorem 2, p. 94] and [10, Theorem 6, p. 127]. Clearly, Q violates the conditions of Theorem D of I. However, by a recent result of Collins [1], Q satisfies the conditions of Theorem B of this paper.

In the following, we shall use the notation of I. In particular, all groups are assumed to be finite. Moreover, if G is a p-group, then $d_e(G)$ is the maximum of the orders of the elementary Abelian subgroups of G and $J_e(G)$ is the subgroup of G generated by the elementary Abelian subgroups of order $d_e(G)$. We shall often use without reference the following two facts: If G is a group and $H \trianglelefteq G$, then

$$O_{2'}(H) \subseteq O_{2'}(G), \qquad O_2(H) \subseteq O_2(G), \qquad \text{and} \qquad O_{2',2}(H) \subseteq O_{2',2}(G).$$

If S is a Sylow p-subgroup of G and E is a strongly closed subgroup of S with respect to G, then for every p-subgroup T of G that contains E, E is strongly closed in T with respect to G.

2. Required Results

The main tools that allow us to work with $O_{2',2}(G)$ rather than $Z^*(G)$ are the following important results of Goldschmidt:

THEOREM G1 [5]. *Suppose S is a Sylow 2-subgroup of a finite group G and A is an Abelian subgroup of S which is strongly closed in S with respect to G. Furthermore, suppose b is an involution in $C_S(A) - A$ such that $A \subseteq O_{2',2}(C(a))$ for every involution a in the coset Ab. Then $A \subseteq O_{2',2}(G)$.*

THEOREM G2 [6, Theorem 3.1]. *Suppose B is an Abelian 2-subgroup of a finite group G and the minimal number of generators of B is at least 3. Suppose also that A is a noncyclic subgroup of B such that $A \subseteq O_{2',2}(C(b))$ for each involution b of B. Let $W = \langle O_{2'}(C(a)) \mid a \in \Omega_1(A) - \{1\}\rangle$. Then W has odd order.*

The two main results of I which we require are the following, which are Theorem B, Theorem 3.1, and Corollary of Theorem D of I, respectively. (In Theorem I 2, $J(S)$ is defined as in [7, p. 271]; however, in this paper, we will not need to apply the case in which $J = J(S)$.)

THEOREM I 1. *Let p be a prime, S be a Sylow p-subgroup of a finite group G, and K be $C_S(\Omega_1(Z(J_e(S))))$. Suppose k is a positive integer and $T_1, \dots, T_k$ are isomorphic subgroups of K. Let $T = \langle T_1, \dots, T_k \rangle$. Assume that:*

(B1) *every element of $N(K)$ permutes $T_1, \dots, T_k$ by conjugation;*

(B2) $T = T_1 \times \cdots \times T_k$;

(B3) *whenever $1 \leqslant i \leqslant k$ and P is a p-subgroup of $N(T_i)$ that contains T_i, then T_i is a direct factor of P and $\Omega_1(T_i) \trianglelefteq N(P) \cap N(T_i')$;*

(B4) *T_1 is not Abelian and $\Omega_1(T_1) \subseteq Z(T_1)$; and*

(B5) *either p is odd or T_1 is not a generalized quaternion group.*

Then $\Omega_1(T)$ is strongly closed in S with respect to G and, if $p = 2$,

$$\Omega_1(T) \cap Z(N(K)) \subseteq Z^*(G).$$

THEOREM I 2. *Suppose p is a prime, S is a Sylow p-subgroup of a group G, and $S = Q \times R$. Assume that Q is indecomposable and not Abelian and that $\Omega_1(Q) \subseteq Z(Q)$. Define J to be $J(S)$ if $p = 2$ and Q is a generalized quaternion group and to be $J_e(S)$ otherwise. Let $K = C_S(\Omega_1(Z(J)))$, $L = N(K) \cap N(Q')$, and $k = | N(K) : L |$. Then there exist isomorphic subgroups $T_1, \dots, T_k \subseteq K$ such that*

(a) $S = T_1 \times R$,

(b) *$T_1, \dots, T_k$ satisfy conditions* (B1), (B2), *and* (B3) *of Theorem* I 1, *and*

(c) *if $Q \cap Z(N(S) \cap N(Q')) \neq 1$, then there exist elements $\tau_1 \in T_1, \dots, \tau_k \in T_k$, each of order p, such that $\tau_1 \cdots \tau_k \in Z(N(K))$.*

THEOREM I 3. *Suppose G is a finite group and some Sylow 2-subgroup S of G has a generalized quaternion group as a direct factor. Then $S \cap Z^*(G) \neq 1$.*

From Theorems I 1 and I 2 we derive the following result.

COROLLARY I 4. *Let S be a Sylow 2-subgroup of a group G. Suppose Q is a non-Abelian, indecomposable direct factor of S and $\Omega_1(Q) \subseteq Z(Q)$. Assume that Q is not a generalized quaternion group. Take $R \subseteq S$ such that $S = Q \times R$. Then there exist $k \geqslant 1$ and Q^*, E, $E_1, \dots, E_k \subseteq S$ such that*

(a) $S = Q^* \times R$,

(b) $E_1 = \Omega_1(Q^*)$,

(c) $E = E_1 \times \cdots \times E_k$,

(d) *E is strongly closed in S with respect to G, and*

(e) *$N(E)$ permutes $E_1, \ldots, E_k$ by conjugation.*

Proof. In this situation, the hypothesis of Theorem I 2 is satisfied. Take k, K, and $T_1, \ldots, T_k$ as in Theorem I 2. By deleting some of the groups T_i and making k smaller, if necessary, we may and will assume that $T_1, \ldots, T_k$ are all conjugate to T_1 in $N(K)$. Set $Q^* = T_1$ and $E_i = \Omega_1(T_i)$ for $i = 1, \ldots, k$. Let $E = E_1 \times \cdots \times E_k$. Then (a), (b), and (c) follow immediately, and Theorem I 1 yields (d).

By (a), $Q^* \cong Q$ and $Z(Q^*) \subseteq Z(S)$. Hence,

$$E_1 = \Omega_1(Q^*) \subseteq Z(Q^*) \subseteq Z(S).$$

As $E_1 \subseteq T_1 \subseteq K$, $E_1 \subseteq Z(K)$. Suppose $1 \leqslant i \leqslant k$. Since T_i is conjugate to T_1 in $N(K)$, $E_i \subseteq Z(K)$. Hence, $E \subseteq Z(K)$, and $K \subseteq C_S(E)$.

By (d), $S \subseteq N(E)$. Therefore, by the Frattini argument,

$$N(E) = C(E)\, N(C_S(E)). \tag{2.1}$$

By a short argument (or Lemma 3.3 of I), K is weakly closed in S with respect to G. Hence, $N(C_S(E)) \subseteq N(K)$. Now, (e) follows from condition (B1) of Theorem I 1 and from (2.1).

In order to use the strong closure of E, we use the following result.

THEOREM S [3, Theorem 6.1]. *Let S be a Sylow p-subgroup of a group G. Suppose E is an Abelian strongly closed subgroup of S with respect to G. Then E controls strong fusion in S with respect to G, that is, whenever $W \subseteq S$, $g \in G$, and $W^g \subseteq S$, then there exist $c \in C(W)$ and $n \in N(E)$ such that $cn = g$.*

Finally, we state a number of lemmas.

LEMMA 1 (Proposition 2.2 of I). *Assume that G is a group, $H_1, \ldots, H_s$ are indecomposable subgroups of G, and $G = H_1 \times \cdots \times H_s$. Suppose*

$$1 \leqslant r \leqslant s, \quad H_1 \cong \cdots \cong H_r, \quad \text{and} \quad H_i \not\cong H_1 \quad \text{for} \quad i = r+1, \ldots, s.$$

Then each automorphism of G permutes the subgroups $H_1Z(G), \ldots, H_rZ(G)$ and permutes the subgroups $H_1', \ldots, H_r'$ in the same manner.

LEMMA 2 (Proposition 2.6 of I). *Let S be a Sylow p-subgroup of a group G.*

Suppose Q is a non-Abelian indecomposable subgroup of S and $S = Q \times R$. Then there exist Q^*, $R^* \subseteq S$ such that

(a) $S = Q^* \times R^* = Q \times R^* = Q^* \times R$,

(b) $Q^*, R^* \trianglelefteq N(S) \cap N(Q') = N(S) \cap N(QZ(S))$, *and*

(c) $Q^{*\prime} = Q'$, $R^{*\prime} = R'$.

LEMMA 3. *Suppose U is a normal 2-subgroup of a group G, and $O_{2'}(C_G(U)) = 1$. Then $O_{2'}(C_G(U)U/U) = 1$.*

Proof. Let $W = C_G(U)$ and $V/U = O_{2'}(WU/U)$. Then V is a 2-solvable group and $U = O_2(V)$. Since $V \trianglelefteq G$, $O_{2'}(V) \subseteq O_{2'}(G) \subseteq O_{2'}(W) = 1$. By a result of Hall and Higman [7, Theorem 6.3.2, pp. 226–228],

$$W \cap V \subseteq C_V(O_2(V)) \subseteq O_2(V) = U.$$

As $U \subseteq V \subseteq UW$, $V = U(V \cap W) = U$. So, $V/U = 1$.

LEMMA 4. *Let S be a Sylow 2-subgroup of a group G. Suppose T and U are subgroups of S and H is a normal subgroup of odd order in G. Let $V = U \cap O_{2',2}(G)$ and $W = U \cap O_{2',2}(C(T))$. For each subgroup L of G, let $\bar{L} = LH/H$. Then*

(a) *the natural mapping of S onto $\bar{S}$ is an isomorphism,*

(b) $C_{\bar{G}}(\bar{T}) = C_G(T)H/H$, $N_{\bar{G}}(\bar{T}) = N_G(T)H/H$,

(c) $N_{\bar{G}}(C_{\bar{S}}(\bar{T})) = N_G(C_S(T))H/H$,

(d) $\bar{V} = \bar{U} \cap O_{2',2}(\bar{G})$ *and* $\bar{W} = \bar{U} \cap O_{2',2}(C_{\bar{G}}(\bar{T}))$, *and*

(e) *if* $H \subseteq L \subseteq G$, *then* $O_{2',2}(L/H) = O_{2',2}(L)H/H$.

Proof. (a) This is obvious, since $S \cap H = 1$.

(b) Take N, $N_0 \subseteq G$ such that $N \supseteq N_0 \supseteq H$, $N/H = N_{\bar{G}}(\bar{T})$, and $N_0/H = C_{\bar{G}}(\bar{T})$. Clearly,

$$C_G(T)H \subseteq N_G(T)H \subseteq N, \qquad TH \trianglelefteq N, \qquad \text{and} \qquad C_{\bar{G}}(\bar{T}) \subseteq N/H.$$

By the Frattini argument,

$$N = (HT)\,N_N(T) = HN_N(T) = HN_G(T).$$

Hence $N_0 = H(N_G(T) \cap N_0) = HC_G(T)$.

(c) This follows from (b).

(d), (e) Part (e) is obvious. By (e),

$$\bar{V} \subseteq \bar{U} \cap O_{2',2}(\bar{G}) = (UH \cap O_{2',2}(G))/H = H(U \cap O_{2',2}(G))/H$$
$$= HV/H = \bar{V}.$$

A similar argument, using (b), completes the proof of (d) and, thus, of the lemma.

3. Strongly Closed Abelian Subgroups of Sylow 2-groups

Theorem 1. *Suppose S is a Sylow 2-subgroup of a group G and E is a strongly closed Abelian subgroup of S with respect to G.*
Suppose $R \subseteq C_S(E)$, $E \cap R = 1$, and $R \trianglelefteq N_G(C_S(E))$. Then

$$E \cap O_{2',2}(G) = E \cap O_{2',2}(C_G(R)).$$

Proof. We use induction on the order of G. Let $F = E \cap O_{2',2}(C_G(R))$. Clearly, $F \supseteq E \cap O_{2',2}(G)$. Hence, it suffices to prove that $F \subseteq O_{2',2}(G)$.
As $S \subseteq N(E)$, the Frattini argument and the hypothesis yield that

$$N_G(E) = C_G(E)\, N_G(C_S(E)) = C_G(E)(N_G(E) \cap N_G(R)). \tag{3.1}$$

Since $F \subseteq E$ and $C_G(R) \trianglelefteq N(R)$,

$$F \trianglelefteq C(E) \qquad \text{and} \qquad F = E \cap O_{2',2}(C(R)) \trianglelefteq N(E) \cap N(R).$$

Hence, $F \trianglelefteq N(E)$. By Theorem S,

(3.2) F is strongly closed in S with respect to G.

Suppose $O_{2'}(G) \neq 1$. Then we are done, by Lemma 4 and induction. Therefore, we will assume, henceforth, that $O_{2'}(G) = 1$.
Let $T = O_2(G)$ and $U = C_T(E)$. Take any $g \in G$. Then $U^g \subseteq T \subseteq S$. By Theorem S, there exists $n \in N(E)$ such that $U^n = U^g$. Hence,

$$U^g \subseteq C(E) \cap T = U.$$

Thus, $U \trianglelefteq G$. Consequently, $E \subseteq C(U) \trianglelefteq G$ and $C_S(U)$ is a Sylow 2-subgroup of $C(U)$. Therefore, $C_S(U)\, C_S(E)$ is a Sylow 2-subgroup of $C(U)\, C_S(E)$.
Suppose $C(U)\, C_S(E) \subset G$. Then, by induction, $F \subseteq O_{2',2}(C(U)\, C_S(E))$. Since $E \subseteq C(U)$,

$$F \subseteq O_{2',2}(C(U)) \subseteq O_{2',2}(G),$$

as desired.
Assume that $C(U)\, C_S(E) = G$. Suppose now that $R \subseteq U$. Then R is centralized by $C(U)$ and normalized by $C_S(E)$. Therefore, $R \trianglelefteq G$ and

$$F \subseteq O_{2',2}(C(R)) \subseteq O_{2',2}(N(R)) = O_{2',2}(G),$$

as desired.

Assume that $C(U)\, C_S(E) = G$ and $R \not\subseteq U$. Since $U = C_T(E)$, $R \cap T = R \cap U \subset R$. For every element g of G and every subgroup L of G, let

$$\bar{g} = gU \qquad \text{and} \qquad \bar{L} = LU/U.$$

Take $x \in R - U$ and $H \subseteq G$ such that $\bar{x} \in \Omega_1(Z(\bar{S})) \cap \bar{R}$, $H \supseteq U$, and $H/U = C_{\bar{G}}(\bar{x})$. Since $U = C_T(E)$, $R \cap T = R \cap U$. Therefore, $\bar{x} \notin \bar{T} = O_2(\bar{G})$. Hence, $H \subset G$. By induction,

$$F \subseteq O_{2',2}(H) \qquad \text{and} \qquad \bar{F} \subseteq O_{2',2}(\bar{H}).$$

Suppose $\bar{f}\bar{x} \in \bar{S} - \bar{T}$ for every $\bar{f} \neq 1$ in $\Omega_1(\bar{F})$. Then (3.2) and Theorem G1 yield that $\bar{F} \subseteq O_{2',2}(\bar{G})$. As $F \subseteq C(U) \trianglelefteq G$,

$$\bar{F} \subseteq O_{2',2}(C(U)U/U) = O_2(C(U)U/U) \subseteq O_2(\bar{G}),$$

by Lemma 3. Since U is a 2-group, $F \subseteq O_2(G) \subseteq O_{2',2}(G)$.

Thus, we are left with the case in which there exists some $f \in F$ for which $\bar{f} \in \Omega_1(\bar{F})$, $\bar{f} \neq 1$, and $fx \in T$. Here, we will derive a contradiction. Since E is Abelian, $fx \in C_T(E) = U$. Take any $g \in N(E) \cap N(R)$. Take $c \in C(U)$ and $d \in C_S(E)$ such that $cd = g$. Then x^g, $x^d \in R$, $xf = fx$, and

$$x^g f^g = (xf)^g = (xf)^{cd} = (xf)^d = x^d f^d = x^d f.$$

So $(x^d)^{-1} x^g = f(f^g)^{-1}$. Since $E \cap R = 1$, $f^g = f$. Thus, $N(E) \cap N(R)$ centralizes f. By (3.1), $N(E)$ centralizes f. By Theorem S, f is weakly closed in S with respect to G. By Corollary 1 of [2], $f \in Z^*(G)$. Since $O_{2'}(G) = 1$ and $f \in E$,

$$f \in O_2(G) \cap C(E) = C_T(E) = U.$$

Thus, $\bar{f} = 1$. This contradicts the choice of f and completes the proof of Theorem 1.

THEOREM 2. *Suppose S is a Sylow 2-subgroup of a group G and E is a strongly closed Abelian subgroup of S with respect to G.*

Suppose $E = E_1 \times \cdots \times E_k$ and $N(E)$ permutes $E_1, \ldots, E_k$ by conjugation (not necessarily transitively). Assume that $k > 1$ and that $|\Omega_1(E_1)| > 2$. Then

$$E_1 \cap O_{2',2}(G) = E_1 \cap O_{2',2}(C(E_2 \cdots E_k)).$$

Proof. We use induction on the sum of $|G|$ and k. Let

$$F_1 = E_1 \cap O_{2',2}(C(E_2 \cdots E_k)).$$

Clearly, $F_1 \supseteq E_1 \cap O_{2',2}(G)$. Hence, it suffices to prove that $F_1 \subseteq O_{2',2}(G)$. By Lemma 4 and induction, we may and will assume that $O_{2'}(G) = 1$.

Assume first that, for some i, E_i is not conjugate to E_1 under $N(E)$. For convenience in notation, we will assume that for some $j < k$ the conjugates of E_1 under $N(E)$ are $E_1, \dots, E_j$. Then $E_1 \cdots E_j$ and $E_{j+1} \cdots E_k$ are both normalized by $N(E)$ and, by Theorem S, are both strongly closed subgroups of S with respect to G. Let $G^* = C(E_{j+1} \cdots E_k)$. By Theorem 1,

$$E_1 \cap O_{2',2}(G) = E_1 \cap O_{2',2}(G^*).$$

Thus, we are finished in this case if $j = 1$. If $j > 1$, then by induction

$$E_1 \cap O_{2',2}(G^*) = E_1 \cap O_{2',2}(C_{G^*}(E_2 \cdots E_j)) = F_1,$$

as desired.

Now assume that, for every i, E_i is conjugate to E_1 under $N(E)$. Then $|\Omega_1(E_i)| > 2$ for all i. For each i, let

$$D_i = (E_1 \cdots E_{i-1}) \times (E_{i+1} \cdots E_k) \qquad \text{and} \qquad F_i = E_i \cap O_{2',2}(C(D_i)).$$

Set $F = F_1 \times \cdots \times F_k$. Then the subgroups F_i are permuted transitively under conjugation by $N(E)$, and $F \trianglelefteq N(E)$. By induction,

(3.3) whenever $E \subseteq H \subset G$, then $F \subseteq O_{2',2}(H)$.

Suppose $E_1 \cap O_{2',2}(G) \neq 1$. Since $O_{2'}(G) = 1$, $E_1 \cap O_{2',2}(G) \subseteq O_2(G) \subseteq S$. Let $E^* = E \cap O_2(G)$. By the strong closure of E in S, it follows that $E^* \trianglelefteq G$. Since $N(E)$ permutes the subgroups E_i transitively by conjugation,

$$C(E^*) \subseteq C(E_1 \cap O_2(G)) \subset G.$$

By (3.3), $F_1 \subseteq F \subseteq O_{2',2}(C(E^*)) \subseteq O_{2',2}(G)$.

Henceforth, we will assume that $F_1 \neq 1$ and that

$$E_1 \cap O_{2',2}(G) = 1. \tag{3.4}$$

Eventually, we will obtain a contradiction.

Let $F_1{}^*$ be a minimal normal subgroup of $N(E_1)$ contained in F_1. Then $F_1{}^*$ is an elementary Abelian group. Define F^* to be the product of the groups $(F_1{}^*)^g$, $g \in N(E)$. Note that there are precisely k such groups $(F_1{}^*)^g$ and that F^* is their direct product. Moreover, $F^* \trianglelefteq N(E)$ and, by Theorem S, F^* is a strongly closed subgroup of S with respect to G. Suppose that $F_1{}^* \subset \Omega_1(E_1)$. Take $e \in \Omega_1(E_1) - F_1{}^*$. Then by (3.4) and Theorem G1, there exists $f \in F^*$ such that $F^* \not\subseteq O_{2',2}(C(ef))$. By (3.3), $C(ef) = G$. However,

$$C_{N(E)}(ef) \subseteq N_{N(E)}(\langle ef, F^* \rangle) = N_{N(E)}(\langle e, F^* \rangle) \subseteq N_{N(E)}(E_1) \subset N(E).$$

This contradiction shows that $F_1{}^* = \Omega_1(E_1)$. Thus, $F^* = \Omega_1(E)$ and $N(E_1)$ acts irreducibly on $\Omega_1(E_1)$. Since $|\Omega_1(E_1)| > 2$, $E_1 \cap Z(N(E_1)) = 1$. By Theorem S,

$$E_1 \cap Z(N(E) \cap N(E_1)) = 1.$$

Since all the subgroups E_i are conjugate under $N(E)$,

$$E_i \cap Z(N(E) \cap N(E_i)) = 1 \qquad \text{for each } i. \tag{3.5}$$

Take any $e \neq 1$ in F^*. Then there exist $e_i \in E_i$ for each i such that $e_1 \cdots e_k = e$. Take j such that $e_j \neq 1$. Then

$$C_{N(E_j) \cap N(E)}(e) \subseteq C_{N(E_j) \cap N(E)}(e_j) \subset N(E_j) \cap N(E),$$

by (3.5). Therefore, $C_G(e) \subset G$. Thus, by (3.3),

$$F^* \subseteq F \subseteq O_{2',2}(C(e)) \qquad \text{for every } e \neq 1 \text{ in } F^*. \tag{3.6}$$

Define W to be the subgroup of G generated by the groups $O_{2'}(C(e))$, $e \in F^* - \{1\}$. Since

$$|F^*| = |\Omega_1(E)| = |\Omega_1(E_1)|^k \geqslant 4^2 = 2^4,$$

(3.6) and Theorem G 2 yield that W has odd order. Define M to be $N(W)$ if $W \neq 1$ and to be $N(F^*)$ if $W = 1$. Then $M \supseteq N(F^*)$. Since $O_{2'}(G) = 1$, (3.4) yields that $M \subset G$.

Take $e \in F^* - \{1\}$. Let T be a Sylow 2-subgroup of $C(e)$ that contains F^*. By Sylow's theorem and the strong closure of F^* in S, F^* is strongly closed in T with respect to G. Hence, by (3.6) and the Frattini argument,

$$\begin{aligned} C(e) &= O_{2',2}(C(e))\, N_{C(e)}(T \cap O_{2',2}(C(e))) \\ &= O_{2'}(C(e))\, N_{C(e)}(F^*) \subseteq W N_G(F^*) \subseteq M. \end{aligned}$$

Thus,

$$C(e) \subseteq M \qquad \text{for every } e \in F^* - \{1\}. \tag{3.7}$$

Suppose $e \in F^* - \{1\}$, $g \in G$, and $e^g \in M$. Then there exists $m \in M$ such that $(e^g)^m \in S$. By the strong closure of F^* in S, $e^{gm} \in E$ and $e^{gm} = e^n$ for some $n \in N(E)$. Then $gmn^{-1} \in C(e) \subseteq M$, by (3.7). Since $m \in M$ and $n \in N(E) \subseteq M$, $g \in M$. This shows that

(3.8) if $e \in F^* - \{1\}$, $g \in G$, and $e^g \in M$, then $g \in M$.

Now take $e \in F_1{}^* - \{1\}$ and $f \in F_2{}^* - \{1\}$. Then e and ef are not conjugate under $N(E)$ and, therefore, are not conjugate in G, by Theorem S. Take $g \in G - M$. By (3.8),

$$(ef)^g \notin M. \tag{3.9}$$

Let

$$D = \langle e, (ef)^g \rangle \qquad \text{and} \qquad x = e(ef)^g.$$

Since e and ef are not conjugate in G, x has even order.

Let S^* be a Sylow 2-subgroup of D that contains e. Then $|S^*| \geqslant 4$ and there exists $d \in D$ such that $S^* = \langle e, (ef)^{gd} \rangle$. Take $h \in G$ such that $S^{*h} \subseteq S$. By (3.8) and the strong closure of F^* in S,

$$h \in M, \qquad e^h, (ef)^{gdh} \in F^*, \qquad \text{and} \qquad S^{*h} \subseteq F^*. \tag{3.10}$$

Hence e^h centralizes $(ef)^{gdh}$, and e centralizes $(ef)^{gd}$. So, $S^* \subseteq C(e) \subseteq M$. Let y be the unique involution in $\langle x \rangle$. Then $y \in S^* \subseteq M$ and, by (3.7) and (3.10), $D^h \subseteq C(y^h) \subseteq M$ and $h \in M$. Therefore, $(ef)^g \in M$, contrary to (3.9). This contradiction completes the proof of Theorem 2.

THEOREM 3. *Suppose S is a Sylow 2-subgroup of a group G and E is an Abelian strongly closed subgroup of S with respect to G.*

Suppose $k \geqslant 1$, $E = E_1 \times \cdots \times E_k$, and $N(E)$ permutes $E_1, \ldots, E_k$ by conjugation (not necessarily transitively). Assume that $|\Omega_1(E_1)| > 2$ and that there exists $\underset{\sim}{Q} \subseteq S$ such that $E \cap \underset{\sim}{Q}' \neq 1$, $E_1 = E \cap \underset{\sim}{Q}$, and $\underset{\sim}{Q}$ is a direct factor of some Sylow 2-subgroup S^* of $N(E_1)$. Then there exists $R \subseteq S^*$ such that

$$S^* = \underset{\sim}{Q} \times R \qquad \textit{and} \qquad E_1 \cap O_{2',2}(C(R)) = E_1 \cap O_{2',2}(G).$$

Proof. We use induction on the sum of $|G|$ and k. By Theorem S and induction, we may and will assume that $E_1, \ldots, E_k$ are transitively permuted by $N(E)$. By the strong closure of E in S, $N(\underset{\sim}{Q}) \subseteq N(E_1)$. Therefore, replacing S^* by some conjugate of S^* in $N(\underset{\sim}{Q})$ if necessary, we may and will assume that $S^* \supseteq N_S(\underset{\sim}{Q}) \supseteq \underset{\sim}{Q}E$. By the strong closure of E, $E \trianglelefteq S^*$.

Suppose $k = 1$. Then $E = E_1 \subseteq \underset{\sim}{Q}$. By Sylow's theorem, E is strongly closed in S^* with respect to G. Hence,

$$1 \neq E \cap \underset{\sim}{Q}' = E \cap (S^*)' \trianglelefteq N(S^*).$$

For every $g \in N(S^*)$, $1 \neq E \cap \underset{\sim}{Q}' \subseteq \underset{\sim}{Q}' \cap (\underset{\sim}{Q}')^g$. By Lemma 1, $N(S^*) \subseteq N(\underset{\sim}{Q}')$. By Lemma 2, there exists $R \subseteq S^*$ such that $R \trianglelefteq N(S^*)$ and $S^* = \underset{\sim}{Q} \times R$. By Theorem 1,

$$E \cap O_{2',2}(G) = E \cap O_{2',2}(C(R)).$$

Suppose $k > 1$. Let $G^* = N(E_2 \cdots E_k)$. Then $G^* \subset G$ and

$$S^* \subseteq N(E_1) \cap N(E) = N(E) \cap G^*.$$

Since S^* is a Sylow 2-subgroup of $N(E_1)$, S^* is a Sylow 2-subgroup of $N_{G^*}(E_1)$. By induction, there exists $R \subseteq S^*$ such that

$$S^* = Q \times R \qquad \text{and} \qquad E_1 \cap O_{2',2}(C_{G^*}(R)) = E_1 \cap O_{2',2}(G^*).$$

Since $E_1 \subseteq C(E_2 \cdots E_k) \trianglelefteq G^*$, Theorem 2 yields that

$$E_1 \cap O_{2',2}(G) = E_1 \cap O_{2',2}(C(E_2 \cdots E_k)) = E_1 \cap O_{2',2}(G^*).$$

This completes the proof of Theorem 3, since

$$E_1 \cap O_{2',2}(C_{G^*}(R)) \supseteq E_1 \cap O_{2',2}(C_G(R)) \supseteq E_1 \cap O_{2',2}(G).$$

4. Proof of Main Results

We first prove Theorem A. Assume the hypothesis of the theorem. Take k, Q^*, E, and $E_1, \ldots, E_k$ as in Corollary I 4. Then

$$E_1 = \Omega_1(Q^*) \qquad \text{and} \qquad S = Q \times R = Q^* \times R.$$

Let $S^* = S$. By Corollary I 4 and Theorem 3, there exists a subgroup R^* of S such that

$$S = Q^* \times R^* \qquad \text{and} \qquad E_1 \cap O_{2',2}(G) = E_1 \cap O_{2',2}(C(R^*)).$$

By the Krull–Remak–Schmidt theorem [9, Satz 12.3, p. 66] (or Proposition 2.1 of I), $S = Q \times R^*$. Since $C(R^*) \subseteq N(R^*) \subseteq G$,

$$E_1 \cap O_{2',2}(G) = E_1 \cap O_{2',2}(N(R^*)),$$

as desired.

The remainder of this section is devoted to the proof of Theorem B. Clearly, condition (B2) implies condition (B1). Assume now that Q violates (B2). Take G and R such that (B2) is violated by G, R, and Q. Let $S = Q \times R$. Then

$$S \cap O_{2',2}(G) = 1. \tag{4.1}$$

We will show that Q violates (B1).

By (4.1), $S \cap Z^*(G) = 1$. Therefore, Q is not a generalized quaternion group, by Theorem I 3. Take Q^* and R^* as in Theorem A. For convenience in notation, we will assume that $Q = Q^*$ and $R = R^*$. By (4.1) and Theorem A,

$$\Omega_1(Q) \cap O_{2',2}(C(R)) = 1. \tag{4.2}$$

Since $R \trianglelefteq S$ and $C(R) \trianglelefteq N(R)$, $C_S(R)$ is a Sylow 2-subgroup of $C(R)$. As $C_S(R) = Q \times Z(R)$,

(4.3) $Q \times Z(R)$ is a Sylow 2-subgroup of $C(R)$.

Let $L = O_{2',2}(C(R))$. By (4.3),

(4.4) $L \cap QZ(R)$ is a Sylow 2-subgroup of L.

However, $Z(R) \subseteq L \cap QZ(R) \subseteq QZ(R)$. Hence, $L \cap QZ(R) = Z(R) \times (L \cap Q)$. By (4.2), $L \cap \Omega_1(Q) = 1$. Therefore, $L \cap Q = 1$ and, by (4.4),

$$L = O_{2'}(C(R))(L \cap QZ(R)) = O_{2'}(C(R))\, Z(R). \tag{4.5}$$

Let $M = O_{2'}(C(R))$, $H = C(R)/M$, and $U = Z(R)M/M$. By (4.5) and the definition of L,

$$L = MZ(R), \qquad U = O_2(H), \qquad \text{and} \qquad O_2(H/U) = 1.$$

By Lemma 3, $O_{2'}(H/U) = 1$. Consequently, $O_{2',2}(H/U) = 1$. By (4.3), Q is isomorphic to a Sylow 2-subgroup of H/U. Thus, Q violates condition (B1). This completes the proof of Theorem B.

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